

CAPITULATION IN THE ABSOLUTELY ABELIAN EXTENSIONS OF SOME NUMBER FIELDS II

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ABSTRACT. We study the capitulation of 2-ideal classes of an infinite family of imaginary biquadratic number fields consisting of fields $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$, where $i = \sqrt{-1}$ and $q_1 \equiv q_2 \equiv -p \equiv -1 \pmod{4}$ are different primes. For each of the three quadratic extensions \mathbb{K}/\mathbb{k} inside the absolute genus field $\mathbb{k}^{(*)}$ of \mathbb{k} , we compute the capitulation kernel of \mathbb{K}/\mathbb{k} . Then we deduce that each strongly ambiguous class of $\mathbb{k}/\mathbb{Q}(i)$ capitulates already in $\mathbb{k}^{(*)}$.

1. Introduction and Notations

Let k be an algebraic number field and let $\mathbf{Cl}_2(k)$ denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group, $\mathbf{Cl}(k)$, of k . We denote by $k^{(*)}$ the absolute genus field of k , that is the maximal abelian unramified extension of k obtained by composing k and an abelian extension over \mathbb{Q} .

Suppose F is a finite extension of k , then we say that an ideal class of k capitulates in F if it is in the kernel of the homomorphism

$$J_F : \mathbf{Cl}(k) \longrightarrow \mathbf{Cl}(F)$$

induced by extension of ideals from k to F . An important problem in Number Theory is to explicitly determine the kernel of J_F , which is usually called the capitulation kernel.

If F is the relative genus field of a cyclic extension K/k , which we denote by $(K/k)^*$ and that is the maximal unramified extension of K which is obtained by composing K and an abelian extension over k , F. Terada states in [15] that all the ambiguous ideal classes of K/k , which are classes of K fixed under any element of $\text{Gal}(K/k)$, capitulate in $(K/k)^*$.

If F is the absolute genus field of an abelian extension K/\mathbb{Q} , then H. Furuya confirms in [16] that every strongly ambiguous class of K/\mathbb{Q} , that is an ambiguous ideal class containing at least one ideal invariant under $\text{Gal}(K/\mathbb{Q})$, capitulates in F .

In this paper, we construct a family of number fields k for which all the strongly ambiguous classes of $k/\mathbb{Q}(i)$ capitulate in $k^{(*)} \subset (k/\mathbb{Q}(i))^*$.

Let $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ and \mathbb{K} be an unramified quadratic extension of \mathbb{k} that is abelian over \mathbb{Q} . Denote by $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$ the group of the strongly ambiguous classes of $\mathbb{k}/\mathbb{Q}(i)$. In [6], we studied the capitulation problem in the absolutely abelian extensions of \mathbb{k} for $d = 2pq$ and $p \equiv q \equiv 1 \pmod{4}$ are different primes, and in [7], we dealt with the same problem assuming $p \equiv -q \equiv 1 \pmod{4}$. In [9, 10, 11] and under the assumption

2010 *Mathematics Subject Classification.* 11R11, 11R16, 11R20, 11R27, 11R29.

Key words and phrases. absolute and relative genus fields, fundamental systems of units, 2-class group, capitulation, quadratic fields, biquadratic fields, multiquadratic CM-fields.

$\text{Cl}_2(\mathbb{k}) \simeq (2, 2, 2)$, we studied the capitulation problem of the 2-ideal classes of \mathbb{k} in its fourteen unramified extensions, within the first Hilbert 2-class field of \mathbb{k} , and we gave the abelian type invariants of the 2-class groups of these fourteen fields. Additionally we determined the structure of the metabelian Galois group $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ of the second Hilbert 2-class field $\mathbb{k}_2^{(2)}$ of \mathbb{k} .

Let $q_1 \equiv q_2 \equiv -p \equiv -1 \pmod{4}$ be different primes and $d = pq_1q_2$. It is the purpose of the present article to pursue this research project. We will compute the capitulation kernel of \mathbb{K}/\mathbb{k} and we will deduce that $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) \subseteq \ker J_{\mathbb{k}(*)}$. As an application we will determine these kernels when $\text{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$.

Let k be a number field, during this paper, we adopt the following notations:

- κ_K : the capitulation kernel of an unramified extension K/k .
- \mathcal{O}_k : the ring of integers of k .
- E_k : the unit group of \mathcal{O}_k .
- W_k : the group of roots of unity contained in k .
- k^+ : the maximal real subfield of k , if it is a CM-field.
- $Q_k = [E_k : W_k E_{k^+}]$ is Hasse's unit index, if k is a CM-field.
- $q(k/\mathbb{Q}) = [E_k : \prod_{i=1}^s E_{k_i}]$ is the unit index of k , if k is multiquadratic, where k_1, \dots, k_s are the quadratic subfields of k .
- $k^{(*)}$: the absolute genus field of k .
- $\text{Cl}_2(k)$: the 2-class group of k .
- $i = \sqrt{-1}$.
- ϵ_m : the fundamental unit of $\mathbb{Q}(\sqrt{m})$, if $m > 1$ is a square-free integer, that is a generator (modulo the roots of unity) for the unit group of the ring of integers of $\mathbb{Q}(\sqrt{m})$.
- $N(a)$: denotes the absolute norm of a number a , i.e., $N_{k/\mathbb{Q}}(a)$ with $a \in k$.
- $x \pm y$ means $x + y$ or $x - y$ for some numbers x and y .

2. Preliminary results

Let us first collect some results that will be useful in what follows.

Let k_j , $1 \leq j \leq 3$, be the three real quadratic subfields of a biquadratic real number field K_0 and $\epsilon_j > 1$ be the fundamental unit of k_j . Since

$$\alpha^2 N_{K_0/\mathbb{Q}}(\alpha) = \prod_{j=1}^3 N_{K_0/k_j}(\alpha)$$

for any $\alpha \in K_0$, the square of any unit of K_0 is in the group generated by the ϵ_j 's, $1 \leq j \leq 3$. Hence, to determine a fundamental system of units of K_0 it suffices to determine which of the units in $B := \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1\epsilon_2, \epsilon_1\epsilon_3, \epsilon_2\epsilon_3, \epsilon_1\epsilon_2\epsilon_3\}$ are squares in K_0 (for details see [18] or [20]).

Lemma 2.1 ([18]). *A fundamental system of units of K_0 consists of three positive units chosen among*

$$B' := B \cup \{\sqrt{\eta} \mid \eta \in B \text{ and } \sqrt{\eta} \in K_0\}.$$

Lemma 2.2 ([20]). *The units $\epsilon \in B$ that can be squares in K_0 are as follows:*

1. $\epsilon = \epsilon_j$ and $N(\epsilon_j) = 1$ with $1 \leq j \leq 3$,

2. $\epsilon = \epsilon_j \epsilon_l$ and $N(\epsilon_j) = N(\epsilon_l) = 1$ with $1 \leq j \neq l \leq 3$,
3. $\epsilon = \epsilon_1 \epsilon_2 \epsilon_3$ and $N(\epsilon_1) = N(\epsilon_2) = N(\epsilon_3)$.

Put $K = K_0(i)$, then to determine a fundamental system of units of K , we will use the following result that the second author has deduced from a theorem of Hasse [17, §21, Satz 15].

Lemma 2.3. [2, p.18]. *Let $n \geq 2$ be an integer and ξ_n a 2^n -th primitive root of unity, then*

$$\xi_n = \frac{1}{2}(\mu_n + \lambda_n i), \quad \text{where} \quad \mu_n = \sqrt{2 + \mu_{n-1}}, \quad \lambda_n = \sqrt{2 - \mu_{n-1}},$$

$$\mu_2 = 0, \lambda_2 = 2 \quad \text{and} \quad \mu_3 = \lambda_3 = \sqrt{2}.$$

Let n_0 be the greatest integer such that ξ_{n_0} is contained in K , $\{\epsilon'_1, \epsilon'_2, \epsilon'_3\}$ a fundamental system of units of K_0 and ϵ a unit of K_0 such that $(2 + \mu_{n_0})\epsilon$ is a square in K_0 (if it exists). Then a fundamental system of units of K is one of the following systems:

1. $\{\epsilon'_1, \epsilon'_2, \epsilon'_3\}$ if ϵ does not exist,
2. $\{\epsilon'_1, \epsilon'_2, \sqrt{\xi_{n_0}\epsilon}\}$ if ϵ exists; in this case $\epsilon = \epsilon'_1{}^{i_1} \epsilon'_2{}^{i_2} \epsilon'_3$, where $i_1, i_2 \in \{0, 1\}$ (up to a permutation).

Lemma 2.4 ([1, Lemma 5]). *Let $d > 1$ be a square-free integer and $\epsilon_d = x + y\sqrt{d}$, where x, y are integers or semi-integers. If $N(\epsilon_d) = 1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$ and $2d(x-1)$ are not squares in \mathbb{Q} .*

Lemma 2.5 ([1, Lemma 6]). *Let $q \equiv -1 \pmod{4}$ be a prime and $\epsilon_q = x + y\sqrt{q}$ be the fundamental unit of $\mathbb{Q}(\sqrt{q})$. Then x is an even integer, $x \pm 1$ is a square in \mathbb{N} and $2\epsilon_q$ is a square in $\mathbb{Q}(\sqrt{q})$.*

Lemma 2.6 ([2], 3.(1) p.19). *Let $d > 2$ be a square-free integer and $k = \mathbb{Q}(\sqrt{d}, i)$, put $\epsilon_d = x + y\sqrt{d}$.*

1. *If $N(\epsilon_d) = -1$, then $\{\epsilon_d\}$ is a fundamental system of units of k .*
2. *If $N(\epsilon_d) = 1$, then $\{\sqrt{i\epsilon_d}\}$ is a fundamental system of units of k if and only if $x \pm 1$ is a square in \mathbb{N} i.e. $2\epsilon_d$ is a square in $\mathbb{Q}(\sqrt{d})$. Else $\{\epsilon_d\}$ is a fundamental system of units of k .*

This result is also in [21].

Lemma 2.7 ([5]). *Let $d \equiv 1 \pmod{4}$ be a positive square free integer and $\epsilon_d = x + y\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Assume $N(\epsilon_d) = 1$, then*

1. *$x+1$ and $x-1$ are not squares in \mathbb{N} i.e. $2\epsilon_d$ is not a square in $\mathbb{Q}(\sqrt{d})$.*
2. *For all prime p dividing d , $p(x+1)$ and $p(x-1)$ are not squares in \mathbb{N} .*

3. Fundamental system of units of some CM-fields

As $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$, so \mathbb{k} admits three unramified quadratic extensions that are abelian over \mathbb{Q} , which are $\mathbb{K}_1 = \mathbb{k}(\sqrt{p}) = \mathbb{Q}(\sqrt{p}, \sqrt{q_1q_2}, i)$, $\mathbb{K}_2 = \mathbb{k}(\sqrt{q_1}) = \mathbb{Q}(\sqrt{q_1}, \sqrt{q_2p}, i)$ and $\mathbb{K}_3 = \mathbb{k}(\sqrt{q_2}) = \mathbb{Q}(\sqrt{q_2}, \sqrt{q_1p}, i)$. Put $\epsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$. In what follows, we determine the fundamental system of units of \mathbb{K}_j , $1 \leq j \leq 3$.

3.1. Fundamental system of units of the field \mathbb{K}_1 .

We begin by determining the systems of fundamental units of \mathbb{K}_1^+ and \mathbb{K}_1 .

Proposition 3.1. *Keep the previous notations. Then $Q_{\mathbb{K}_1} = 1$ and*

1. *If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_p, \epsilon_{q_1 q_2}, \sqrt{\epsilon_{pq_1 q_2}}\}$ is a fundamental system of units of both of \mathbb{K}_1^+ and \mathbb{K}_1 .*
2. *Else $\{\epsilon_p, \epsilon_{q_1 q_2}, \sqrt{\epsilon_{q_1 q_2} \epsilon_{pq_1 q_2}}\}$ is a fundamental system of units of both of \mathbb{K}_1^+ and \mathbb{K}_1 .*

Proof. As $N(\epsilon_p) = -1$, then by Lemma 2.2 only $\epsilon_{q_1 q_2}$, $\epsilon_{pq_1 q_2}$ and $\epsilon_{q_1 q_2} \epsilon_{pq_1 q_2}$ can be squares in \mathbb{K}_1^+ .

Put $\epsilon_{q_1 q_2} = a + b\sqrt{q_1 q_2}$, then $a^2 - 1 = b^2 q_1 q_2$. Hence by Lemmas 2.4 and 2.7 we get that only the number $2q_1(a \pm 1)$ (i.e. $2q_2(a \pm 1)$) is a square in \mathbb{N} . So there exist b_1 and b_2 in \mathbb{Z} such that

$$\begin{cases} a \pm 1 &= 2b_1^2 q_1 \\ a \mp 1 &= 2b_2^2 q_2, \end{cases}$$

therefore $\sqrt{\epsilon_{q_1 q_2}} = b_1 \sqrt{q_1} + b_2 \sqrt{q_2}$, which implies that $q_1 \epsilon_{q_1 q_2}$ and $q_2 \epsilon_{q_1 q_2}$ are squares in \mathbb{K}_1^+ but $\epsilon_{q_1 q_2}$ is not.

Since $N(\epsilon_{pq_1 q_2}) = 1$, then $x^2 - 1 = y^2 pq_1 q_2$. Hence Lemmas 2.4 and 2.7 allowed us to distinguish the following cases:

- a. If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{pq_1 q_2}} = y_1 \sqrt{p} + y_2 \sqrt{q_1 q_2}$, hence $\epsilon_{pq_1 q_2}$ is a square in \mathbb{K}_1^+ .
- b. If $2q_1(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{pq_1 q_2}} = y_1 \sqrt{q_1} + y_2 \sqrt{pq_2}$, hence $q_1 \epsilon_{pq_1 q_2}$ and $pq_2 \epsilon_{pq_1 q_2}$ are squares in \mathbb{K}_1^+ but $\epsilon_{pq_1 q_2}$ is not.
- c. If $2q_2(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{pq_1 q_2}} = y_1 \sqrt{q_2} + y_2 \sqrt{pq_1}$, hence $q_2 \epsilon_{pq_1 q_2}$ and $pq_1 \epsilon_{pq_1 q_2}$ are squares in \mathbb{K}_1^+ but $\epsilon_{pq_1 q_2}$ is not.

Consequently, we have

1. If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\epsilon_{pq_1 q_2}$ is a square in \mathbb{K}_1^+ . Thus Lemmas 2.1 and 2.3 yield that $\{\epsilon_p, \epsilon_{q_1 q_2}, \sqrt{\epsilon_{pq_1 q_2}}\}$ is a fundamental system of units of both of \mathbb{K}_1^+ and \mathbb{K}_1 .
2. If $2q_1(x \pm 1)$ or $2q_2(x \pm 1)$ is a square in \mathbb{N} , then $q_1 \epsilon_{pq_1 q_2}$ or $q_2 \epsilon_{pq_1 q_2}$ is a square in \mathbb{K}_1^+ . As $q_1 \epsilon_{q_1 q_2}$ and $q_2 \epsilon_{q_1 q_2}$ are squares in \mathbb{K}_1^+ , so $\epsilon_{q_1 q_2} \epsilon_{pq_1 q_2}$ is a square in \mathbb{K}_1^+ . Thus Lemmas 2.1 and 2.3 yield that $\{\epsilon_p, \epsilon_{q_1 q_2}, \sqrt{\epsilon_{q_1 q_2} \epsilon_{pq_1 q_2}}\}$ is a fundamental system of units of both of \mathbb{K}_1^+ and \mathbb{K}_1 .

□

3.2. Fundamental system of units of the field \mathbb{K}_2 .

Let us now determine the fundamental system of units's of $\mathbb{K}_2^+ = \mathbb{Q}(\sqrt{q_1}, \sqrt{pq_2})$ and $\mathbb{K}_2 = \mathbb{Q}(\sqrt{q_1}, \sqrt{pq_2}, i)$.

Proposition 3.2. *Keep the previous notations and put $\epsilon_{pq_2} = a + b\sqrt{pq_2}$. Then $Q_{\mathbb{K}_2} = 2$. Moreover we have:*

1. *Assume $2q_1(x \pm 1)$ is a square in \mathbb{N} , then*
 - i. *If $a \pm 1$ is a square in \mathbb{N} , then $\{\epsilon_{q_1}, \sqrt{\epsilon_{q_1} \epsilon_{pq_2}}, \sqrt{\epsilon_{pq_1 q_2}}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and that of \mathbb{K}_2 is $\{\sqrt{\epsilon_{q_1} \epsilon_{pq_2}}, \sqrt{\epsilon_{pq_1 q_2}}, \sqrt{i \epsilon_{q_1}}\}$.*

- ii. Else $\{\epsilon_{q_1}, \epsilon_{pq_2}, \sqrt{\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and that of \mathbb{K}_2 is $\{\epsilon_{pq_2}, \sqrt{\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_1}}\}$.
- 2. Assume $2q_1(x \pm 1)$ is not a square in \mathbb{N} , then
 - i. If $a \pm 1$ is a square in \mathbb{N} , then $\{\epsilon_{q_1}, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}}, \epsilon_{pq_1q_2}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and that of \mathbb{K}_2 is $\{\sqrt{\epsilon_{q_1}\epsilon_{pq_2}}, \epsilon_{pq_1q_2}, \sqrt{i\epsilon_{q_1}}\}$.
 - ii. If $p(a \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_{q_1}, \epsilon_{pq_2}, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and that of \mathbb{K}_2 is $\{\epsilon_{pq_2}, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_1}}\}$.
 - iii. If $2p(a \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_{q_1}, \epsilon_{pq_2}, \sqrt{\epsilon_{pq_2}\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and that of \mathbb{K}_2 is $\{\epsilon_{pq_2}, \sqrt{\epsilon_{pq_2}\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_1}}\}$.

Proof. By Lemma 2.2 the units that can be squares in \mathbb{K}_2 are: $\epsilon_{q_1}, \epsilon_{pq_2}, \epsilon_{pq_1q_2}, \epsilon_{q_1}\epsilon_{pq_2}, \epsilon_{q_1}\epsilon_{pq_1q_2}, \epsilon_{pq_1}\epsilon_{pq_1q_2}$ and $\epsilon_{q_1}\epsilon_{pq_2}\epsilon_{pq_1q_2}$.

According to Lemma 2.5, $2\epsilon_{q_1}$ is a square in \mathbb{K}_2^+ but ϵ_{q_1} is not.

Put $\epsilon_{pq_2} = a + b\sqrt{pq_2}$, then $a^2 - 1 = b^2pq_2$. Hence Lemma 2.4 allowed us to distinguish the following cases:

- a. If $a \pm 1$ is a square in \mathbb{N} , then there exist b_1 and b_2 in \mathbb{Z} such that

$$\begin{cases} a \pm 1 &= b_1^2, \\ a \mp 1 &= b_2^2pq_2, \end{cases}$$

thus $\sqrt{2\epsilon_{pq_2}} = b_1 + b_2\sqrt{pq_2}$. Therefore $2\epsilon_{pq_2}$ a square in \mathbb{K}_1^+ but ϵ_{pq_2} is not.

- b. If $p(a \pm 1)$ is a square in \mathbb{N} , then there exist b_1 and b_2 in \mathbb{Z} such that

$$\begin{cases} a \pm 1 &= b_1^2p, \\ a \mp 1 &= b_2^2q_2, \end{cases}$$

thus $\sqrt{2\epsilon_{pq_2}} = b_1\sqrt{p} + b_2\sqrt{q_2}$. Therefore $2p\epsilon_{pq_2}$ and $2q_2\epsilon_{q_1q_2}$ are squares in \mathbb{K}_2^+ but ϵ_{pq_2} and $2\epsilon_{pq_2}$ are not.

- c. If $2p(a \pm 1)$ is a square in \mathbb{N} , then there exist b_1 and b_2 in \mathbb{Z} such

$$\begin{cases} a \pm 1 &= 2b_1^2p, \\ a \mp 1 &= 2b_2^2q_2, \end{cases}$$

thus $\sqrt{\epsilon_{pq_2}} = b_1\sqrt{p} + b_2\sqrt{q_2}$. Therefore $p\epsilon_{pq_2}$ and $q_2\epsilon_{pq_2}$ are squares in \mathbb{K}_2^+ but ϵ_{pq_2} is not.

As $N(\epsilon_{pq_1q_2}) = 1$, then $x^2 - 1 = y^2pq_1q_2$; hence Lemmas 2.4 and 2.7 allowed us to distinguish the following cases:

- a'. If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{pq_1q_2}} = y_1\sqrt{p} + y_2\sqrt{q_1q_2}$, thus $p\epsilon_{pq_1q_2}$ and $q_1q_2\epsilon_{pq_1q_2}$ are squares in \mathbb{K}_2^+ but $\epsilon_{pq_1q_2}$ is not.
- b'. If $2q_1(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{pq_1q_2}} = y_1\sqrt{q_1} + y_2\sqrt{pq_2}$, thus $\epsilon_{pq_1q_2}$ is a square in \mathbb{K}_2^+ .
- c'. If $2q_2(x \pm 1)$ is a square in \mathbb{N} , then $\sqrt{\epsilon_{pq_1q_2}} = y_1\sqrt{q_2} + y_2\sqrt{pq_1}$, thus $q_2\epsilon_{pq_1q_2}$ and $pq_1\epsilon_{pq_1q_2}$ are squares in \mathbb{K}_1^+ but $\epsilon_{pq_1q_2}$ is not.

Consequently, we have

- 1. Assume $2q_1(x \pm 1)$ is a square in \mathbb{N} , then $\epsilon_{pq_1q_2}$ is a square in \mathbb{K}_2^+ .
 - i. If $a \pm 1$ is a square in \mathbb{N} , then $2\epsilon_{pq_2}$ is a square in \mathbb{K}_2^+ ; thus $\epsilon_{q_1}\epsilon_{pq_2}$ is a square in \mathbb{K}_2^+ , since $2\epsilon_{q_1}$ is. Therefore, by Lemma 2.1 $\{\epsilon_{q_1}, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}}, \sqrt{\epsilon_{pq_1q_2}}\}$ is a fundamental

- system of units of \mathbb{K}_2^+ , and according to Lemma 2.3 $\{\sqrt{\epsilon_{q_1}\epsilon_{pq_2}}, \sqrt{\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_1}}\}$ is a fundamental system of units of \mathbb{K}_2 .
- ii. Else $\epsilon_{pq_1q_2}$ will be a square in \mathbb{K}_2^+ ; hence by Lemma 2.1 $\{\epsilon_{q_1}, \epsilon_{pq_2}, \sqrt{\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and according to Lemma 2.3 $\{\epsilon_{pq_2}, \sqrt{\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_1}}\}$ is a fundamental system of units of \mathbb{K}_2 .
2. Assume $2q_1(x \pm 1)$ is not a square in \mathbb{N} , then $\epsilon_{pq_1q_2}$ is not a square in \mathbb{K}_2^+ .
- i. If $a \pm 1$ is a square in \mathbb{N} , then $2\epsilon_{pq_2}$ is a square in \mathbb{K}_2^+ ; hence $\epsilon_{q_1}\epsilon_{pq_2}$ is a square in \mathbb{K}_2^+ , since $2\epsilon_{q_1}$ is a square in \mathbb{N} . Thus by Lemma 2.1 $\{\epsilon_{q_1}, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}}, \epsilon_{pq_1q_2}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and according to Lemma 2.3 $\{\sqrt{\epsilon_{q_1}\epsilon_{pq_2}}, \epsilon_{pq_1q_2}, \sqrt{i\epsilon_{q_1}}\}$ is a fundamental system of units of \mathbb{K}_2 .
- ii. If $p(a \pm 1)$ is a square in \mathbb{N} , then $2p\epsilon_{pq_2}$ and $2q_2\epsilon_{pq_2}$ are squares in \mathbb{K}_2^+ . On the other hand, we have $p\epsilon_{pq_1q_2}$ or $q_2\epsilon_{pq_1q_2}$ is a square in \mathbb{K}_1^+ , thus $\epsilon_{q_1}\epsilon_{pq_2}\epsilon_{pq_1q_2}$ is a square in \mathbb{K}_2^+ , since $2\epsilon_{q_1}$ is a square in \mathbb{N} . Therefore by Lemma 2.1 $\{\epsilon_{q_1}, \epsilon_{pq_2}, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_2^+ , and according to Lemma 2.3 $\{\epsilon_{pq_2}, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_1}}\}$ is a fundamental system of units of \mathbb{K}_2 .
- iii. The last case is treated similarly. □

3.3. Fundamental system of units of the field \mathbb{K}_3 .

Since q_1 and q_2 play symmetrical roles, then the fundamental system of units's of $\mathbb{K}_3^+ = \mathbb{Q}(\sqrt{q_2}, \sqrt{pq_1})$ and $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q_2}, \sqrt{pq_1}, i)$ are easily deduced.

Proposition 3.3. *Keep the previous notations and put $\epsilon_{pq_1} = a + b\sqrt{pq_1}$. Then $Q_{\mathbb{K}_3} = 2$. Moreover we have.*

- 1.) Assume $2q_2(x \pm 1)$ is a square in \mathbb{N} , then
- i. If $a \pm 1$ is a square in \mathbb{N} , then $\{\epsilon_{q_2}, \sqrt{\epsilon_{q_2}\epsilon_{pq_1}}, \sqrt{\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_3^+ , and that of \mathbb{K}_3 is $\{\sqrt{\epsilon_{q_2}\epsilon_{pq_1}}, \sqrt{\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_2}}\}$.
- ii. Else $\{\epsilon_{q_2}, \epsilon_{pq_1}, \sqrt{\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_3^+ , and that of \mathbb{K}_3 is $\{\epsilon_{pq_1}, \sqrt{\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_2}}\}$.
- 2.) Assume $2q_2(x \pm 1)$ is not a square in \mathbb{N} , then
- i. If $a \pm 1$ is a square in \mathbb{N} , then $\{\epsilon_{q_2}, \sqrt{\epsilon_{q_2}\epsilon_{pq_1}}, \epsilon_{pq_1q_2}\}$ is a fundamental system of units of \mathbb{K}_3^+ , and that of \mathbb{K}_3 is $\{\sqrt{\epsilon_{q_2}\epsilon_{pq_1}}, \epsilon_{pq_1q_2}, \sqrt{i\epsilon_{q_2}}\}$.
- ii. If $p(a \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_{q_2}, \epsilon_{pq_1}, \sqrt{\epsilon_{q_2}\epsilon_{pq_1}\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_3^+ , and that of \mathbb{K}_3 is $\{\epsilon_{pq_1}, \sqrt{\epsilon_{q_2}\epsilon_{pq_1}\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_2}}\}$.
- iii. If $2p(a \pm 1)$ is a square in \mathbb{N} , then $\{\epsilon_{q_2}, \epsilon_{pq_1}, \sqrt{\epsilon_{pq_1}\epsilon_{pq_1q_2}}\}$ is a fundamental system of units of \mathbb{K}_3^+ , and that of \mathbb{K}_3 is $\{\epsilon_{pq_1}, \sqrt{\epsilon_{pq_1}\epsilon_{pq_1q_2}}, \sqrt{i\epsilon_{q_2}}\}$.

4. The ambiguous classes of $\mathbb{k}/\mathbb{Q}(i)$

Let $F = \mathbb{Q}(i)$ and $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$. We denote by $Am(\mathbb{k}/F)$ the group of the ambiguous classes of \mathbb{k}/F and by $Am_s(\mathbb{k}/F)$ the subgroup of $Am(\mathbb{k}/F)$ generated by the strongly ambiguous classes. As $p \equiv 1 \pmod{4}$, so there exist e and f in \mathbb{N} such that $p = e^2 + 4f^2 = \pi_1\pi_2$. Put $\pi_1 = e + 2if$ and $\pi_2 = e - 2if$. Let \mathcal{H}_j (resp. \mathcal{Q}_j) be the prime ideal of \mathbb{k} above π_j (resp. q_j), where $j \in \{1, 2\}$. It is easy to see that $\mathcal{H}_j^2 = (\pi_j)$ and $\mathcal{Q}_j^2 = (q_j)$. Therefore $[\mathcal{Q}_j]$ and $[\mathcal{H}_j]$ are in $Am_s(\mathbb{k}/F)$, for all $j \in \{1, 2\}$.

Keep the notation $\epsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$. In this section, we will determine generators of $\text{Am}_s(\mathbb{k}/F)$ and $\text{Am}(\mathbb{k}/F)$. Let us first prove the following result.

Lemma 4.1. *Consider the prime ideals \mathcal{H}_j and \mathcal{Q}_j of \mathbb{k} , $1 \leq j \leq 2$.*

1. *If $2p(x \pm 1)$ is a square in \mathbb{N} , then $|\langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle| = 4$.*
2. *Else $|\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle| = 4$*

Proof. Since $\mathcal{H}_j^2 = (\pi_j)$, $1 \leq j \leq 2$, and since also $\sqrt{e^2 + (2f)^2} = \sqrt{p} \notin \mathbb{Q}(\sqrt{pq_1q_2})$, so, according to [4, Proposition 1], \mathcal{H}_j are not principal in \mathbb{k} .

1. If $2p(x \pm 1)$ is a square in \mathbb{N} , and since $(\mathcal{H}_1\mathcal{H}_2)^2 = (p_1)$, $\mathcal{Q}_j^2 = (q_j)$ and $(\mathcal{H}_1\mathcal{Q}_j)^2 = (q_j)$, hence by [4, Proposition 2 and Remark 1], $\mathcal{H}_1\mathcal{H}_2$ is principal in \mathbb{k} and \mathcal{Q}_j , $\mathcal{H}_1\mathcal{Q}_j$ are not. Thus the result.

2. If $2p(x \pm 1)$ is not a square in \mathbb{N} , i.e. $2q_1(x \pm 1)$ or $2q_2(x \pm 1)$ is a square in \mathbb{N} ; then $\mathcal{H}_1\mathcal{H}_2$ is not principal in \mathbb{k} and \mathcal{Q}_1 or \mathcal{Q}_2 is (by [4, Proposition 2]). On the other hand, if \mathcal{Q}_1 (resp. \mathcal{Q}_2) is principal, then $[\mathcal{H}_1\mathcal{H}_2] = [\mathcal{Q}_2]$ (resp. $[\mathcal{H}_1\mathcal{H}_2] = [\mathcal{Q}_1]$). \square

Determine now generators of $\text{Am}_s(\mathbb{k}/F)$ and $\text{Am}(\mathbb{k}/F)$. According to the ambiguous class number formula ([12]), the genus number, $[(\mathbb{k}/F)^* : \mathbb{k}]$, is given by:

$$|\text{Am}(\mathbb{k}/F)| = [(\mathbb{k}/F)^* : \mathbb{k}] = \frac{h(F)2^{t-1}}{[E_F : E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times)]}, \quad (1)$$

where $h(F)$ is the class number of F and t is the number of finite and infinite primes of F ramified in \mathbb{k}/F . Moreover as the class number of F is equal to 1, so the formula (1) yields that

$$|\text{Am}(\mathbb{k}/F)| = [(\mathbb{k}/F)^* : \mathbb{k}] = 2^r, \quad (2)$$

where $r = \text{rank Cl}_2(\mathbb{k}) = t - e - 1$ and $2^e = [E_F : E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times)]$ (see for example [22]). The relation between $|\text{Am}(\mathbb{k}/F)|$ and $|\text{Am}_s(\mathbb{k}/F)|$ is given by the following formula (see for example [13]):

$$\frac{|\text{Am}(\mathbb{k}/F)|}{|\text{Am}_s(\mathbb{k}/F)|} = [E_F \cap N_{\mathbb{k}/F}(\mathbb{k}^\times) : N_{\mathbb{k}/F}(E_{\mathbb{k}})]. \quad (3)$$

To continue, we need the following lemma.

Lemma 4.2. *Let $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$ be different primes, $F = \mathbb{Q}(i)$ and $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$.*

1. *If $p \equiv 1 \pmod{8}$, then i is a norm in \mathbb{k}/F .*
2. *If $p \equiv 5 \pmod{8}$, then i is not a norm in \mathbb{k}/F .*

Proof. We proceed as in Lemma 11 of [7]. \square

Proposition 4.3. *Let $(\mathbb{k}/F)^*$ denote the relative genus field of \mathbb{k}/F . Then*

1.
 - i. *If $p \equiv 1 \pmod{8}$, then $\mathbb{k}^{(*)} \subsetneq (\mathbb{k}/F)^*$ and $[(\mathbb{k}/F)^* : \mathbb{k}^{(*)}] = 2$.*
 - ii. *Else $\mathbb{k}^{(*)} = (\mathbb{k}/F)^*$.*
2. *Assume $p \equiv 1 \pmod{8}$.*
 - i. *If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle$.*
 - ii. *Else, $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.*

iii. there exist an unambiguous ideal \mathcal{I} in $\mathbb{k}/\mathbb{Q}(i)$ of order 2 such that

$$\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \begin{cases} \langle [\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}] \rangle, & \text{if } 2p(x \pm 1) \text{ is a square in } \mathbb{N}, \\ \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}] \rangle, & \text{otherwise.} \end{cases}$$

3. Assume $p \equiv 5 \pmod{8}$, then

$$\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \begin{cases} \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle, & \text{if } 2p(x \pm 1) \text{ is a square in } \mathbb{N}, \\ \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle, & \text{otherwise.} \end{cases}$$

Proof. 1. As $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$, so $[\mathbb{k}^{(*)} : \mathbb{k}] = 4$. Moreover, according to [22, Proposition 2, p. 90], $r = \text{rank Cl}_2(\mathbb{k}) = 3$ if $p \equiv 1 \pmod{8}$ and $r = \text{rank Cl}_2(\mathbb{k}) = 2$ if $p \equiv 5 \pmod{8}$, so $[(\mathbb{k}/F)^* : \mathbb{k}] = 4$ or 8 . Hence $[(\mathbb{k}/F)^* : \mathbb{k}^{(*)}] = 1$ or 2 , and the results derived.

2. Note first that, by Lemma 2.7, $x+1$ and $x-1$ are never squares in \mathbb{N} . Thus from Lemma 2.6 we get $E_{\mathbb{k}} = \langle i, \epsilon_{pq_1q_2} \rangle$.

Assume $p \equiv 1 \pmod{8}$, hence i is a norm in $\mathbb{k}/\mathbb{Q}(i)$ (Lemma 4.2), thus Formula (3) yields that

$$\frac{|\text{Am}(\mathbb{k}/\mathbb{Q}(i))|}{|\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))|} = [E_{\mathbb{Q}(i)} \cap N_{\mathbb{k}/\mathbb{Q}(i)}(\mathbb{k}^\times) : N_{\mathbb{k}/\mathbb{Q}(i)}(E_{\mathbb{k}})] = 2$$

since $[E_{\mathbb{Q}(i)} \cap N_{\mathbb{k}/\mathbb{Q}(i)}(\mathbb{k}^\times) : N_{\mathbb{k}/\mathbb{Q}(i)}(E_{\mathbb{k}})] = [\langle i \rangle : \langle -1 \rangle] = 2$.

On the other hand, as $p \equiv 1 \pmod{8}$, we have just shown that $r = 3$. Therefore $|\text{Am}(\mathbb{k}/\mathbb{Q}(i))| = 2^4$ and thus $|\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))| = 4$

i. If $2p(x \pm 1)$ is a square in \mathbb{N} which is equivalent to $2q_1q_2(x \pm 1)$ is a square in \mathbb{N} , then $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = 2\text{Am}_s(\mathbb{k}/\mathbb{Q}(i))$, hence by Lemma 4.1 we get

$$\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle.$$

ii. If $2q_1(x \pm 1)$ or $2q_2(x \pm 1)$ is a square in \mathbb{N} , then Lemma 4.1 yields that

$$\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle.$$

Consequently, in the two cases there exists an unambiguous ideal \mathcal{I} in \mathbb{k}/F of order 2 such that

$$\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \begin{cases} \langle [\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}] \rangle, & \text{if } 2p(x \pm 1) \text{ is a square in } \mathbb{N}, \\ \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}] \rangle, & \text{else.} \end{cases}$$

By Chebotarev theorem, \mathcal{I} can always be chosen as a prime ideal of \mathbb{k} above a prime ℓ in \mathbb{Q} , which splits completely in \mathbb{k} .

3. Assume $p \equiv 5 \pmod{8}$, hence i is not a norm in $\mathbb{k}/\mathbb{Q}(i)$ (Lemma 4.2). Proceeding similarly as in 2., we get

$$\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \begin{cases} \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle, & \text{if } 2p(x \pm 1) \text{ is a square in } \mathbb{N}, \\ \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle, & \text{else.} \end{cases}$$

This completes the proof. \square

5. Capitulation

Let p , q_1 and q_2 be primes satisfying $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$. Set $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$ and denote by $\mathbb{k}^{(*)}$ the genus field of \mathbb{k} , then $\mathbb{k}^{(*)} = \mathbb{Q}(\sqrt{p}, \sqrt{q_1}, \sqrt{q_2}, i)$. The unramified quadratic extensions of \mathbb{k} , abelian over \mathbb{Q} , are $\mathbb{K}_1 = \mathbb{k}(\sqrt{p}) = \mathbb{Q}(\sqrt{p}, \sqrt{q_1q_2}, i)$, $\mathbb{K}_2 = \mathbb{k}(\sqrt{q_1}) = \mathbb{Q}(\sqrt{q_1}, \sqrt{pq_2}, i)$ and $\mathbb{K}_3 = \mathbb{k}(\sqrt{q_2}) = \mathbb{Q}(\sqrt{q_2}, \sqrt{pq_1}, i)$. Keep the notations $\epsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$ denoting the fundamental unit of $\mathbb{Q}(\sqrt{pq_1q_2})$ and $p = e^2 + 4f^2 = \pi_1\pi_2$, where $\pi_1 = e + 2if$, $\pi_2 = e - 2if$. Let $Q_{\mathbb{k}}$ be the unit index of \mathbb{k} , and \mathcal{H}_j be the ideal of \mathbb{k} lies above π_j . Denote also by \mathcal{Q}_j the prime ideal of \mathbb{k} above q_j , $j = 1, 2$.

In this section, we will determine the classes of $\mathbf{Cl}_2(\mathbb{k})$, the 2-class group of \mathbb{k} , that capitulate in \mathbb{K}_j , for all $j \in \{1, 2, 3\}$. For this we need the following theorem.

Theorem 5.1 ([14]). *Let K/k be a cyclic extension of prime degree, then the number of classes that capitulate in K/k is: $[K:k][E_k : N_{K/k}(E_K)]$, where E_k and E_K are the unit groups of k and K respectively.*

5.1. The number of classes capitulating in each \mathbb{K}_j .

Recall that $\kappa_{\mathbb{K}_j}$ denotes the capitulation kernel of the unramified extension \mathbb{K}_j/\mathbb{k} .

Theorem 5.2. *Let \mathbb{K}_j , $1 \leq j \leq 3$, be the three unramified quadratic extensions of \mathbb{k} defined above. Then*

1. $|\kappa_{\mathbb{K}_1}| = 4$.
2. Let $\epsilon_{pq_2} = a + b\sqrt{pq_2}$, then
 - i. If $a \pm 1$ is a square in \mathbb{N} and $2q_1(x+1)$, $2q_1(x-1)$ are not, then $|\kappa_{\mathbb{K}_2}| = 4$.
 - ii. In the other cases $|\kappa_{\mathbb{K}_2}| = 2$.
3. Let $\epsilon_{pq_1} = a + b\sqrt{pq_1}$, then
 - i. If $a \pm 1$ is a square in \mathbb{N} and $2q_2(x+1)$, $2q_2(x-1)$ are not, then $|\kappa_{\mathbb{K}_3}| = 4$.
 - ii. In the other cases $|\kappa_{\mathbb{K}_3}| = 2$.

Proof. Note first that, according to Lemma 2.7, $x+1$ and $x-1$ are never squares in \mathbb{N} , hence by Lemma 2.6, $E_{\mathbb{k}} = \langle i, \epsilon_{pq_1q_2} \rangle$.

1. By Proposition 3.1 we have $E_{\mathbb{K}_1} = \langle i, \epsilon_p, \epsilon_{q_1q_2}, \sqrt{\epsilon_{pq_1q_2}} \rangle$ or $E_{\mathbb{K}_1} = \langle i, \epsilon_p, \epsilon_{q_1q_2}, \sqrt{\epsilon_{q_1q_2}\epsilon_{pq_1q_2}} \rangle$, hence $N_{\mathbb{K}_1/\mathbb{k}}(E_{\mathbb{K}_1}) = \langle -1, \epsilon_{pq_1q_2} \rangle$. Thus $[E_{\mathbb{k}} : N_{\mathbb{K}_1/\mathbb{k}}(E_{\mathbb{K}_1})] = 2$. Therefore Theorem 5.1 implies that $|\kappa_{\mathbb{K}_1}| = 4$.
2. i. If $a \pm 1$ is a square in \mathbb{N} and $2q_1(x+1)$, $2q_1(x-1)$ are not, then Proposition 3.2(2)(i) yields that $N_{\mathbb{K}_2/\mathbb{k}}(E_{\mathbb{K}_2}) = \langle i, \epsilon_{pq_1q_2}^2 \rangle$, hence $[E_{\mathbb{k}} : N_{\mathbb{K}_2/\mathbb{k}}(E_{\mathbb{K}_2})] = 2$. Thus Theorem 5.1 implies that $|\kappa_{\mathbb{K}_2}| = 4$.
- ii. The other cases are grouped together in Proposition 3.2 (assertions 1, 2), then $N_{\mathbb{K}_2/\mathbb{k}}(E_{\mathbb{K}_2}) = \langle i, \epsilon_{pq_1q_2} \rangle$. Thus $[E_{\mathbb{k}} : N_{\mathbb{K}_2/\mathbb{k}}(E_{\mathbb{K}_2})] = 1$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_2}| = 2$.
3. This point is similarly treated.

□

5.2. Capitulation in \mathbb{K}_1 .

Theorem 5.3. *Let p , q_1 and q_2 be different primes such that $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$. Put $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$, $\mathbb{K}_1 = \mathbb{Q}(\sqrt{p}, \sqrt{q_1q_2}, i)$ and $\epsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$, then*

1. If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle$.
2. Else, $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.

Proof. We have already shown, in Lemma 4.1, that \mathcal{H}_1 , \mathcal{H}_2 , \mathcal{Q}_j and $\mathcal{H}_k \mathcal{Q}_j$, $j, k = 1, 2$, are not principal in \mathbb{k} . On the other hand, by Proposition 6.3 of [8] \mathcal{H}_1 and \mathcal{H}_2 capitulate in \mathbb{K}_1 .

1. If $2p(x \pm 1)$ is a square in \mathbb{N} , then by [4, Proposition 2] $\mathcal{H}_1 \mathcal{H}_2$ is principal in \mathbb{k} , i.e. $[\mathcal{H}_1] = [\mathcal{H}_2]$. The proof of the Proposition 3.1, allows us to conclude that $q_1 \epsilon_{q_1 q_2}$ and $q_2 \epsilon_{q_1 q_2}$ are squares in \mathbb{K}_1 ; hence there exists $\gamma \in \mathbb{K}_1$ such that $\mathcal{Q}_1^2 = (\gamma^2)$. Thus $\mathcal{Q}_1 = (\gamma)$, so the result.

2. If $2p(x + 1)$ and $2p(x - 1)$ are not squares in \mathbb{N} , then $\mathcal{H}_1 \mathcal{H}_2$ is not principal in \mathbb{k} ; which yields the result. \square

Numerical Examples 5.4.

1. The case where $2p(x \pm 1)$ is a square in \mathbb{N} .

$d = p.q_1.q_2$	$2p(x + 1)$	$2p(x - 1)$	$\mathcal{H}_1 \mathcal{H}_2$ in \mathbb{k}	\mathcal{Q}_1 in \mathbb{k}	\mathcal{H}_1	\mathcal{Q}_1
$105 = 5.3.7$	420	$400 = 20^2$	$[0, 0]$	$[2, 0]$	$[0, 0]$	$[0, 0]$
$345 = 5.23.3$	67620	$67600 = 260^2$	$[0, 0]$	$[2, 0]$	$[0, 0]$	$[0, 0]$
$357 = 17.3.7$	357	$289 = 17^2$	$[0, 0, 0]$	$[1, 1, 0]$	$[0, 0]$	$[0, 0]$
$561 = 17.11.3$	17774724	$17774656 = 4216^2$	$[0, 0, 0]$	$[0, 0, 1]$	$[0, 0, 0]$	$[0, 0, 0]$
$645 = 5.3.43$	645	$625 = 25^2$	$[0, 0]$	$[4, 0]$	$[0, 0]$	$[0, 0]$
$705 = 5.47.3$	2371620	$2371600 = 1540^2$	$[0, 0]$	$[6, 0]$	$[0, 0]$	$[0, 0]$
$805 = 5.7.23$	7245	$7225 = 85^2$	$[0, 0]$	$[4, 0]$	$[0, 0]$	$[0, 0]$

2. The case where $2p(x + 1)$ and $2p(x - 1)$ are not squares in \mathbb{N} .

$d = p.q_1.q_2$	$2p(x + 1)$	$2p(x - 1)$	$\mathcal{H}_1 \mathcal{H}_2$ in \mathbb{k}	\mathcal{H}_1	\mathcal{H}_2
$165 = 5.3.11$	75	55	$[2, 0]$	$[0, 0]$	$[0, 0]$
$273 = 13.7.3$	18928	18876	$[2, 0]$	$[0, 0]$	$[0, 0]$
$285 = 5.3.19$	95	75	$[4, 0]$	$[0, 0]$	$[0, 0]$
$429 = 13.11.3$	1911	1859	$[4, 0]$	$[0, 0]$	$[0, 0]$
$465 = 5.3.31$	158720	158700	$[4, 0]$	$[0, 0]$	$[0, 0]$
$609 = 29.7.3$	35130368	35130252	$[4, 0]$	$[0, 0]$	$[0, 0]$
$665 = 5.7.19$	137200	137180	$[6, 0]$	$[0, 0]$	$[0, 0]$
$741 = 13.19.3$	3211	3159	$[6, 0]$	$[0, 0]$	$[0, 0]$
$1533 = 73.3.7$	37303	37011	$[3, 1, 0]$	$[0, 0, 0]$	$[0, 0, 0]$

5.3. Capitulation in \mathbb{K}_2 .

Let p , q_1 and q_2 be different primes such that $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$. Put $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1 q_2}, i)$, $\mathbb{K}_2 = \mathbb{Q}(\sqrt{q_1}, \sqrt{pq_2}, i)$ and $\epsilon_{pq_2} = a + b\sqrt{pq_2}$.

Lemma 5.5. *If $a \pm 1$ is a square in \mathbb{N} , then $p \equiv 1 \pmod{8}$.*

Proof. If $a \pm 1$ is a square in \mathbb{N} , then $\begin{cases} a \pm 1 = y_1^2, \\ a \mp 1 = pq_2 y_2^2. \end{cases}$

Hence $1 = \left(\frac{a \pm 1}{p}\right) = \left(\frac{a \mp 1 \pm 2}{p}\right) = \left(\frac{2}{p}\right)$. \square

Therefore, if we suppose that $a \pm 1$ is a square in \mathbb{N} , then from Proposition 4.3 we get:

- i. If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle$.

- ii. Else, $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.
- iii. there exists an unambiguous ideal \mathcal{I} in $\mathbb{k}/\mathbb{Q}(i)$ of order 2 such that

$$\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \begin{cases} \langle [\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}] \rangle, & \text{if } 2p(x \pm 1) \text{ is a square in } \mathbb{N}, \\ \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}] \rangle, & \text{otherwise.} \end{cases}$$

The ideal \mathcal{I} can be constructed by using the result:

Lemma 5.6 ([19]). *Let p_1, p_2, \dots, p_n be distinct primes and for each j , let $e_j = \pm 1$. Then there exist infinitely many primes ℓ such that $\left(\frac{p_j}{\ell}\right) = e_j$, for all j .*

Let ℓ be a prime congruent to 1 (mod 4) and satisfying $\left(\frac{pq_1q_2}{\ell}\right) = -\left(\frac{q_1}{\ell}\right) = 1$, thus ℓ splits completely in \mathbb{k} . Therefore \mathcal{I} is one of the ideals of \mathbb{k} above ℓ ; since $\left(\frac{q_1}{\ell}\right) = -1$, so \mathcal{I} remaind inert in \mathbb{K}_2 . We proceed as in [7] to prove that \mathcal{I} , $\mathcal{H}_1\mathcal{I}$, $\mathcal{H}_2\mathcal{I}$ and $\mathcal{H}_1\mathcal{H}_2\mathcal{I}$ or \mathcal{I} , $\mathcal{H}_1\mathcal{I}$, $\mathcal{Q}_1\mathcal{I}$ and $\mathcal{Q}_1\mathcal{H}_1\mathcal{I}$ are not principal in \mathbb{k} .

Theorem 5.7. *Keep the previous hypothesis and notations and put $\epsilon_{pq_2} = a + b\sqrt{pq_2}$, $\epsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$.*

1. *If $a \pm 1$ is a square in \mathbb{N} and $2q_1(x+1)$, $2q_1(x-1)$ are not, then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{Q}_1], [\mathcal{I}] \rangle$ or $\langle [\mathcal{Q}_1], [\mathcal{H}_1\mathcal{I}] \rangle$.*
2. *If $a \pm 1$ and $2q_1(x \pm 1)$ are squares in \mathbb{N} , then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{I}] \rangle$ or $\langle [\mathcal{I}\mathcal{H}_1] \rangle$ or $\langle [\mathcal{I}\mathcal{H}_2] \rangle$ or $\langle [\mathcal{I}\mathcal{H}_1\mathcal{H}_2] \rangle$.*
3. *If $a+1$ and $a-1$ are not squares in \mathbb{N} and $2q_1(x \pm 1)$ is, then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{Q}_2] \rangle = \langle [\mathcal{H}_1\mathcal{H}_2] \rangle$.*
4. *If $a+1$, $a-1$, $2q_1(x+1)$ and $2q_1(x-1)$ are not squares in \mathbb{N} , then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{Q}_1] \rangle$.*

Proof. Let \mathcal{H}_1 , \mathcal{H}_2 , \mathcal{Q}_1 and \mathcal{Q}_2 denote always the ideals of \mathbb{k} above $\pi_1 = e + 2if$, $\pi_2 = e - 2if$, q_1 and q_2 respectively.

1. Suppose $a \pm 1$ is a square in \mathbb{N} and $2q_1(x+1)$, $2q_1(x-1)$ are not. We know according to Proposition 3.2 that $E_{\mathbb{K}_2} = \langle i, \sqrt{\epsilon_{q_1}\epsilon_{pq_2}}, \epsilon_{pq_1q_2}, \sqrt{i\epsilon_{q_1}} \rangle$ and that four classes capitulate in \mathbb{K}_2 one of them is \mathcal{Q}_1 . To proof the result, it suffices to prove that \mathcal{H}_1 does not capitulate in \mathbb{K}_2 .

If \mathcal{H}_1 capitulates in \mathbb{K}_2 , then there exists $\alpha \in \mathbb{K}_2$ such that $\mathcal{H}_1 = (\alpha)$; hence $(\alpha^2) = (\pi_1)$. As a result, there exists a unit $\epsilon \in \mathbb{K}_2$ such that $\pi_1\epsilon = \alpha^2$. The unit ϵ can not be real or purely imaginary. In fact, if it is real (same proof if it is purely imaginary), then by putting $\alpha = \alpha_1 + i\alpha_2$, where α_i are in \mathbb{K}_2^+ , we get $\alpha_1^2 - \alpha_2^2 + 2\alpha_1\alpha_2 = \epsilon(e + 2if)$, thus

$$\begin{cases} \alpha_1^2 - \alpha_2^2 &= e\epsilon, \\ \alpha_1\alpha_2 &= f\epsilon, \end{cases}$$

hence $f\alpha_1^2 - e\alpha_2\alpha_1 - f\alpha_2^2 = 0$. But this implies that $\alpha_1 = \frac{\alpha_2(e \pm \sqrt{p})}{f}$, and thus $\sqrt{p} \in \mathbb{K}_2^+$, which is absurd.

As $\pi_1\epsilon = \alpha^2$, so, by the norm $N_{\mathbb{K}_2/\mathbb{k}}$, we get $\pi_1^2 N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) = N_{\mathbb{K}_2/\mathbb{k}}(\alpha)^2$ with $N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) \in E_{\mathbb{k}} = \langle i, \epsilon_{pq_1q_2} \rangle$. Therefore, we have the following result

$$N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) \in \{\pm 1, \pm i, \pm \epsilon_{pq_1q_2}, \pm i\epsilon_{pq_1q_2}\}.$$

- a. If $N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) = \pm i$, then $\pi_1^2(\pm i) = N_{\mathbb{K}_2/\mathbb{k}}(\alpha)^2$; hence $\sqrt{i} \in \mathbb{k}$, which is absurd.
- b. If $N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) = \pm \epsilon_{pq_1q_2}$, then $\pi_1^2(\pm \epsilon_{pq_1q_2}) = N_{\mathbb{K}_2/\mathbb{k}}(\alpha)^2$; this in turn yields that $\sqrt{\epsilon_{pq_1q_2}} \in \mathbb{k}$, which is absurd.
- c. If $N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) = \pm i\epsilon_{pq_1q_2}$, then $\pi_1^2(\pm i\epsilon_{pq_1q_2}) = N_{\mathbb{K}_2/\mathbb{k}}(\alpha)^2$; this in turn yields that $\sqrt{i\epsilon_{pq_1q_2}} \in \mathbb{k}$, which is absurd.

- d. If $N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) = 1$, then there exist a, b, c and d in $\{0, 1\}$ such that $\epsilon = i^a \sqrt{\epsilon_{q_1} \epsilon_{pq_2}}^b \epsilon_{pq_1 q_2}^c \sqrt{i \epsilon_{q_1}}^d$ and $N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) = 1$, hence $(-1)^a \epsilon_{pq_1 q_2}^{2c} i^d = 1$. Thus obviously we must have $a = c = d = 0$. As a result, we get $\epsilon = \sqrt{\epsilon_{q_1} \epsilon_{pq_2}}^b$ is a real, which is absurd.
- e. If $N_{\mathbb{K}_2/\mathbb{k}}(\epsilon) = -1$, then, by applying the same argument, we get $\epsilon = i \sqrt{\epsilon_{q_1} \epsilon_{pq_2}}^b$, which is purely imaginary, and this is absurd.

To complete the proof of the first point of the corollary, we give examples that affirm the two cases of capitulation:

Numerical Examples 5.8.

$a \pm 1$ is a square in \mathbb{N} and $2q_1(x+1)$, $2q_1(x-1)$ are not.

$d = p.q_1.q_2$	\mathcal{I} in \mathbb{k}	\mathcal{H}_1	\mathcal{I} in \mathbb{K}_2	$\mathcal{H}_1 \mathcal{I}$ in \mathbb{K}_2
$4029 = 17.3.79$	$[5, 1, 1]$	$[170, 0]$	$[170, 0]$	$[0, 0]$
$4029 = 17.79.3$	$[5, 0, 0]$	$[30, 0]$	$[30, 0]$	$[0, 0]$
$4029 = 17.79.3$	$[0, 0, 1]$	$[30, 0]$	$[30, 0]$	$[0, 0]$
$4029 = 17.79.3$	$[5, 0, 0]$	$[30, 0]$	$[30, 0]$	$[0, 0]$
$4029 = 17.79.3$	$[5, 0, 0]$	$[30, 0]$	$[30, 0]$	$[0, 0]$
$4029 = 17.79.3$	$[0, 1, 1]$	$[30, 0]$	$[30, 0]$	$[0, 0]$
$4029 = 17.79.3$	$[0, 0, 1]$	$[30, 0]$	$[30, 0]$	$[0, 0]$
$4029 = 17.79.3$	$[5, 1, 0]$	$[30, 0]$	$[30, 0]$	$[0, 0]$

2. Suppose $a \pm 1$ and $2q_1(x \pm 1)$ are squares in \mathbb{N} ; then according to Proposition 3.2, $2p(x+1)$, $2p(x-1)$, $2q_2(x+1)$ and $2q_2(x-1)$ are not squares in \mathbb{N} ; but $2pq_2(x \pm 1)$ is. Therefore [4, Proposition 2] implies that $\mathcal{H}_1 \mathcal{H}_2$ and \mathcal{Q}_2 are not principal in \mathbb{k} , but \mathcal{Q}_1 and $\mathcal{H}_1 \mathcal{H}_2 \mathcal{Q}_2$ are; hence $[\mathcal{Q}_2] = [\mathcal{H}_1 \mathcal{H}_2]$. Which implies that $\text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}] \rangle$. By using the same method applied in the above point, we show that \mathcal{H}_1 , \mathcal{H}_2 and $[\mathcal{Q}_2] = [\mathcal{H}_1 \mathcal{H}_2]$ do not capitulate in \mathbb{K}_2 . Thus $\kappa_{\mathbb{K}_2}$ consists of one of the following ideal classes: \mathcal{I} , $\mathcal{H}_1 \mathcal{I}$, $\mathcal{H}_2 \mathcal{I}$ and $\mathcal{H}_1 \mathcal{H}_2 \mathcal{I}$. The following examples highlight these statements:

Numerical Examples 5.9.

$a \pm 1$ and $2q_1(x \pm 1)$ are squares in \mathbb{N} .

$d = p.q_1.q_2$	\mathcal{I} in \mathbb{K}_2	$\mathcal{I} \mathcal{H}_1$ in \mathbb{K}_2	$\mathcal{I} \mathcal{H}_2$ in \mathbb{K}_2	$\mathcal{H}_1 \mathcal{H}_2 \mathcal{I}$ in \mathbb{K}_2
$969 = 17.19.3$	$[0, 0, 0, 0]$	$[3, 1, 1, 1]$	$[0, 1, 1, 0]$	$[0, 1, 0, 0]$
$1533 = 73.3.7$	$[0, 0, 1, 0]$	$[0, 0, 0, 0]$	$[21, 1, 0, 1]$	$[21, 1, 0, 0]$
$2037 = 97.3.7$	$[9, 0, 0, 0]$	$[9, 0, 0, 1]$	$[0, 0, 0, 0]$	$[0, 0, 0, 1]$
$2193 = 17.43.3$	$[3, 0, 0, 0]$	$[0, 0, 1, 0]$	$[0, 1, 1, 0]$	$[0, 0, 0, 0]$

3. Suppose $a+1$ and $a-1$ are not squares in \mathbb{N} , and assume $2q_1(x \pm 1)$ is. Then Propositions 1 and 2 of [4] imply that \mathcal{Q}_1 is principal in \mathbb{k} , \mathcal{Q}_2 and $\mathcal{H}_1 \mathcal{H}_2$ are not, and $[\mathcal{Q}_2] = [\mathcal{H}_1 \mathcal{H}_2]$. Moreover, $p(a \pm 1)$ or $2p(a \pm 1)$ is a square in \mathbb{N} , hence $q_2 \epsilon_{pq_2}$ or $2q_2 \epsilon_{pq_2}$ is a square in \mathbb{K}_2 ; and this yields that \mathcal{Q}_2 and $\mathcal{H}_1 \mathcal{H}_2$ capitulate in \mathbb{K}_2 . Here are some examples that illustrate our results.

Numerical Examples 5.10.

$a+1$ and $a-1$ are not squares in \mathbb{N} and $2q_1(x \pm 1)$ is.

$d = p.q_1.q_2$	a	$2q_1(x+1)$	$2q_1(x-1)$	$\mathcal{H}_1\mathcal{H}_2$ in \mathbb{k}	\mathcal{Q}_2 in \mathbb{k}	$\mathcal{H}_1\mathcal{H}_2$	\mathcal{Q}_2
$165 = 5.11.3$	4	165	$121 = 11^2$	$[2, 0]$	$[2, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$273 = 13.3.7$	1574	4368	$4356 = 66^2$	$[2, 0]$	$[2, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$285 = 5.19.3$	4	$361 = 19^2$	285	$[4, 0]$	$[4, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$385 = 5.11.7$	6	$2108304 = 1452^2$	2108260	$[2, 0]$	$[2, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$429 = 13.3.11$	12	$441 = 21^2$	429	$[4, 0]$	$[4, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$465 = 5.31.3$	4	$984064 = 992^2$	983940	$[4, 0]$	$[4, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$609 = 29.7.3$	28	$8479744 = 2912^2$	8479716	$[4, 0]$	$[4, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$665 = 5.19.7$	6	521360	$521284 = 722^2$	$[6, 0]$	$[6, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$741 = 13.3.19$	85292	741	$729 = 27^2$	$[6, 0]$	$[6, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$777 = 37.7.3$	295	$3136 = 56^2$	3108	$[4, 0]$	$[4, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$885 = 5.59.3$	4	14160	$13924 = 118^2$	$[6, 0]$	$[6, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$897 = 13.3.23$	415	$3600 = 60^2$	3588	$[4, 0]$	$[4, 0]$	$[0, 0, 0]$	$[0, 0, 0]$
$1045 = 5.11.19$	39	$1089 = 33^2$	1045	$[4, 0]$	$[4, 0]$	$[0, 0, 0]$	$[0, 0, 0]$

4. If $a+1$, $a-1$, $2q_1(x+1)$ and $2q_1(x-1)$ are not squares in \mathbb{N} , then \mathcal{Q}_1 is not principal in \mathbb{k} ; and as $\sqrt{q_1} \in \mathbb{K}_2$, so \mathcal{Q}_1 capitulate in \mathbb{K}_2 .

Numerical Examples 5.11.

$a+1$, $a-1$, $2q_1(x+1)$ and $2q_1(x-1)$ are not squares in \mathbb{N} .

$d = p.q_1.q_2$	$a+1$	$a-1$	$2q_1(x+1)$	$2q_1(x-1)$	\mathcal{Q}_1 in \mathbb{k}	\mathcal{Q}_1
$105 = 5.7.3$	5	3	588	560	$[2, 0]$	$[0, 0]$
$165 = 5.3.11$	90	88	45	33	$[2, 0]$	$[0, 0]$
$273 = 13.7.3$	26	24	10192	10164	$[2, 0]$	$[0, 0]$
$285 = 5.3.19$	40	38	57	45	$[4, 0]$	$[0, 0]$
$345 = 5.3.23$	1127	1125	40572	40560	$[2, 0]$	$[0, 0]$
$345 = 5.23.3$	5	3	311052	310960	$[2, 0]$	$[0, 0]$
$385 = 5.7.11$	90	88	1341648	1341620	$[2, 0]$	$[0, 0]$
$429 = 13.11.3$	26	24	1617	1573	$[4, 0]$	$[0, 0]$
$465 = 5.3.31$	250	248	95232	95220	$[4, 0]$	$[0, 0]$

□

5.4. Capitulation in \mathbb{K}_3 .

Let p , q_1 and q_2 be different primes satisfying $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$. Put $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$, $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q_2}, \sqrt{pq_1}, i)$ and $\epsilon_{pq_1} = a + b\sqrt{pq_1}$. As q_1 and q_2 play symmetric roles, so the following results are deduced from the above by analogy. Let \mathcal{I} be the ideal defined as above and assume the prime ℓ satisfies the conditions: $\ell \equiv 1 \pmod{4}$ and $\left(\frac{pq_1q_2}{\ell}\right) = -\left(\frac{q_2}{\ell}\right) = 1$.

Theorem 5.12. *Keep the obvious notations and hypothesis. Put $\epsilon_{pq_1} = a + b\sqrt{pq_1}$, then*

1. *If $a \pm 1$ is a square in \mathbb{N} and $2q_2(x+1)$, $2q_2(x-1)$ are not, then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{Q}_2], [\mathcal{I}] \rangle$ or $\langle [\mathcal{Q}_2], [\mathcal{H}_1\mathcal{I}] \rangle$.*
2. *If $a \pm 1$ and $2q_2(x \pm 1)$ are squares in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{I}] \rangle$ or $\langle [\mathcal{I}\mathcal{H}_1] \rangle$ or $\langle [\mathcal{I}\mathcal{H}_2] \rangle$ or $\langle [\mathcal{I}\mathcal{H}_1\mathcal{H}_2] \rangle$.*
3. *If $a+1$ and $a-1$ are not squares in \mathbb{N} and $2q_2(x \pm 1)$ is, then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{Q}_1] \rangle$.*
4. *If $a+1$, $a-1$, $2q_2(x+1)$ and $2q_2(x-1)$ are not squares in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{Q}_2] \rangle$.*

5.5. Capitulation in $\mathbb{k}^{(*)}$.

The following theorem is a simple deduction from Theorems 5.3, 5.7 and 5.12.

Theorem 5.13. *Let p , q_1 and q_2 be different primes satisfying $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$. Put $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$ and denote by $\mathbb{k}^{(*)}$ its genus field. Let $\epsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$ be the fundamental unit of $\mathbb{Q}(\sqrt{pq_1q_2})$.*

1. *Assume $p \equiv 1 \pmod{8}$, then there exists an unambiguous ideal \mathcal{I} of $\mathbb{k}/\mathbb{Q}(i)$ of order 2 such that:*
 - i. *If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\langle [\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}] \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*
 - ii. *Else, $\langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}] \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*
2. *Assume $p \equiv 5 \pmod{8}$.*
 - i. *If $2p(x \pm 1)$ is a square in \mathbb{N} , then $\langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*
 - ii. *Else, $\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle \subseteq \kappa_{\mathbb{k}^{(*)}}$.*

6. Application

Let $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$ be different primes such that $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$. According to [3], $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$ if and only if p , q_1 and q_2 satisfy the following two conditions:

A: $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$ and $\left(\frac{2}{p}\right) = \left(\frac{q_1}{q_2}\right) = -\left(\frac{q_2}{q_1}\right) = 1$.

B: One of the following three conditions is satisfied:

- (I): $\left(\frac{p}{q_1}\right) \left(\frac{p}{q_2}\right) = -1$ and $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = -1$.
- (II): $\left(\frac{p}{q_1}\right) \left(\frac{p}{q_2}\right) = -1$, $\left(\frac{2}{q_1}\right) = 1$ and $\left(\frac{2}{q_2}\right) = -1$.
- (III): $\left(\frac{p}{q_1}\right) = \left(\frac{p}{q_2}\right) = -1$ and $\left(\frac{2}{q_1}\right) \left(\frac{2}{q_2}\right) = -1$.

Remark 6.1. We keep the notations defined in [5, Definition 1], and we add the following definition assuming $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$ satisfying the condition A.

1. p , q_1 and q_2 are said of type $B(III)(1)$ if $\left(\frac{p}{q_1}\right) = \left(\frac{p}{q_2}\right) = -1$ and $-\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = 1$.
2. p , q_1 and q_2 are said of type $B(III)(2)$ if $\left(\frac{p}{q_1}\right) = \left(\frac{p}{q_2}\right) = -1$ and $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = 1$.

To continue we need the following results.

Lemma 6.2. *Let $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$ be different primes satisfying the condition A, and put $\epsilon_{pq_2} = a + b\sqrt{pq_2}$.*

1. *If p , q_1 and q_2 are of type $B(I)$ or $B(II)$, then $a + 1$ is not a square in \mathbb{N} .*
2. *If p , q_1 and q_2 are of type $B(I)(1)$ or $B(II)(1)$, then $p(a - 1)$ and $2p(a + 1)$ are not squares in \mathbb{N} .*
3. *If p , q_1 and q_2 are of type $B(I)(2)$ or $B(II)(2)$, then $p(a + 1)$ and $2p(a - 1)$ are not squares in \mathbb{N} .*
4. *If p , q_1 and q_2 are of type $B(III)(1)$, then $a - 1$ and $p(a + 1)$ are not squares in \mathbb{N} .*
5. *If p , q_1 and q_2 are of type $B(III)(2)$, then $a + 1$ and $p(a - 1)$ are not squares in \mathbb{N} .*
6. *If p , q_1 and q_2 are of type $B(III)$, then $2p(a + 1)$ is not a square in \mathbb{N} .*

Proof. We know that $N(\epsilon_{pq_2}) = 1$, then $a^2 - 1 = b^2pq_2$, hence by Lemma 2.4 and the decomposition uniqueness in \mathbb{Z} there exist b_1, b_2 in \mathbb{Z} such that:

$$(1) \begin{cases} a \pm 1 = b_1^2, \\ a \mp 1 = pq_2 b_2^2; \end{cases} \quad \text{or} \quad (2) \begin{cases} a \pm 1 = pb_1^2, \\ a \mp 1 = q_2 b_2^2; \end{cases} \quad \text{or} \quad (3) \begin{cases} a \pm 1 = 2pb_1^2, \\ a \mp 1 = 2q_2 b_2^2; \end{cases}$$

1. Suppose

$$\begin{cases} a + 1 = b_1^2, \\ a - 1 = pq_2 b_2^2, \end{cases}$$

then $\left(\frac{2}{q_2}\right) = 1$, but this contradicts the conditions $B(I)$ and $B(II)$, hence the result.

The other cases are checked similarly. \square

Remark 6.3. If $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$, then by Proposition 4.3 and [5, Lemma 3], we deduce that:

1. $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle \subsetneq \text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \mathbf{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_1], [\mathcal{Q}_1], [\mathcal{I}] \rangle$, if p, q_1 and q_2 are of type $B(III)$,
2. $\text{Am}_s(\mathbb{k}/\mathbb{Q}(i)) = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle \subsetneq \text{Am}(\mathbb{k}/\mathbb{Q}(i)) = \mathbf{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{I}] \rangle$, otherwise.

Theorem 6.4. Let $p \equiv -q_1 \equiv -q_2 \equiv 1 \pmod{4}$ be different primes such that $\mathbf{Cl}_2(\mathbb{k})$ is of type $(2, 2, 2)$, where $\mathbb{k} = \mathbb{Q}(\sqrt{pq_1q_2}, i)$.

1. Exactly four classes of $\mathbf{Cl}_2(\mathbb{k})$ capitulate in \mathbb{K}_1 .
 - i. If p, q_1 and q_2 are of type $B(III)$, then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{Q}_1] \rangle$.
 - ii. Else, $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.
2. Put $\epsilon_{pq_2} = a + b\sqrt{pq_2}$, then the capitulation in \mathbb{K}_2 is given by:
 - i. If p, q_1 and q_2 are of type $B(I)(1)$ or $B(II)(1)$, then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_1\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_2\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_1\mathcal{H}_2\mathcal{I}] \rangle$.
 - ii. If p, q_1 and q_2 are of type $B(I)(2)$ or $B(II)(2)$, then
 - a. If $a - 1$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1\mathcal{I}] \rangle$.
 - b. Else, $\kappa_{\mathbb{K}_2} = \langle [\mathcal{Q}_1] \rangle = \langle [\mathcal{H}_1\mathcal{H}_2] \rangle$.
 - iii. If p, q_1 and q_2 are of type $B(III)$, then
 - a. If $a \pm 1$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_2} = \langle [\mathcal{Q}_1], [\mathcal{I}] \rangle$ or $\langle [\mathcal{Q}_1], [\mathcal{H}_1\mathcal{I}] \rangle$.
 - b. Else, $\kappa_{\mathbb{K}_2} = \langle [\mathcal{Q}_1] \rangle$.
3. Put $\epsilon_{pq_1} = a + b\sqrt{pq_1}$, then the capitulation in \mathbb{K}_3 is given by:
 - i. If p, q_1 and q_2 are of type $B(I)(2)$ or $B(II)(2)$, then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_1\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_2\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_1\mathcal{H}_2\mathcal{I}] \rangle$.
 - ii. If p, q_1 and q_2 are of type $B(I)(1)$ or $B(II)(1)$, then
 - a. If $a - 1$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1\mathcal{I}] \rangle$.
 - b. Else, $\kappa_{\mathbb{K}_3} = \langle [\mathcal{Q}_2] \rangle = \langle [\mathcal{H}_1\mathcal{H}_2] \rangle$.
 - iii. If p, q_1 and q_2 are of type $B(III)$, then
 - a. If $a \pm 1$ is a square in \mathbb{N} , then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{Q}_2], [\mathcal{I}] \rangle$ or $\langle [\mathcal{Q}_2], [\mathcal{H}_1\mathcal{I}] \rangle$.
 - b. Else, $\kappa_{\mathbb{K}_3} = \langle [\mathcal{Q}_2] \rangle$.

Proof. Let $\epsilon_{pq_1q_2} = x + y\sqrt{pq_1q_2}$ denote the fundamental unit of $\mathbb{Q}(\sqrt{pq_1q_2})$.

1. We know, by [5, Lemma 3], that if p, q_1 and q_2 are of type $B(III)$, then $2p(x-1)$ is a square in \mathbb{N} , and otherwise $2p(x-1)$, $2p(x+1)$ are not squares in \mathbb{N} . Thus Theorem 5.3 implies the results.
2. Put $\epsilon_{pq_2} = a + b\sqrt{pq_2}$.
 - i. Suppose p, q_1 and q_2 satisfy the conditions A and $B(I)(1)$ or $B(II)(1)$, then, by [5, Lemma 3], $2q_1(x+1)$ is a square in \mathbb{N} . On the other hand, from Lemma

- 6.2, $p(a-1)$ and $2p(a+1)$ are not squares in \mathbb{N} , thus $a-1$ is a square in \mathbb{N} . Therefore, we are in the hypotheses of Theorem 5.7(2), thus the results.
- ii. Suppose p , q_1 and q_2 satisfy the conditions A and $B(I)(2)$ or $B(II)(2)$, then, by [5, Lemma 3], $2q_2(x-1)$ is a square in \mathbb{N} i.e. $2pq_1(x+1)$ is a square in \mathbb{N} . Thus [4, Proposition 1] implies that $[\mathcal{H}_1\mathcal{H}_2] = [Q_1]$. On the other hand, from Lemma 6.2, one of the numbers $a-1$, $p(a-1)$ or $2p(a+1)$ is a square in \mathbb{N} . So we are in the hypotheses of Theorem 5.7 (1) or (4), thus the results.
- iii. Suppose p , q_1 and q_2 satisfy the conditions A and $B(III)$, then, by [5, Lemma 3], $2p(x-1)$ is a square in \mathbb{N} , and by Lemma 6.2, $2p(a+1)$ is not a square in \mathbb{N} .
- If p , q_1 and q_2 are of type $B(III)(1)$, then Lemma 6.2 implies that one of the numbers $a+1$, $p(a-1)$ or $2p(a-1)$ is a square in \mathbb{N} .
 - If p , q_1 and q_2 are of type $B(III)(2)$, then Lemma 6.2 implies that one of the numbers $a-1$, $p(a+1)$ or $2p(a-1)$ is a square in \mathbb{N} .
- Therefore,
- a. If $a \pm 1$ is a square in \mathbb{N} , then the result is assured by Theorem 5.7(1).
 - b. Else, the result is assured by Theorem 5.7(4).
3. These results are shown as in 2. □

Corollary 6.5. *Keep the hypotheses and notations mentioned in Theorem 6.4. Then all the classes of $\text{Cl}_2(\mathbb{k})$ capitulate in $\mathbb{k}^{(*)}$ i.e.*

$$\kappa_{\mathbb{k}^{(*)}} = \text{Cl}_2(\mathbb{k}) = \text{Am}(\mathbb{k}/\mathbb{Q}(i)).$$

7. ACKNOWLEDGEMENT

We would like to thank the referee of our paper for his precious remarks and suggestions.

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